More on Ordinary Differential Equations with Laplace Transforms

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Seminar in Engineering Analysis

October 30, 2017

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Outline

- Review last class
  - Definitions of Laplace transforms
  - Getting a transform by integration
  - Finding transforms (and inverse transforms) from tables and theorems
  - Applications to differential equations
- Examples of applications to systems of homogenous and nonhomogeneous equations

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Review Transform Definition

- Transforms from a function of time, f(t), to a function in a complex space, F(s), where s is a complex variable
- The transform of a function, is written as F(s) = ℒf(t) where ℒ denotes the Laplace transform (use ℑ for ℒ in some equations)
- Laplace transform defined as the following integral

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

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Simple Laplace Transforms

f(t)	F(s)	f(t)	F(s)
t^n	$n!/s^{n+1}$	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
t^x	$\Gamma(x+1)/s^{x+1}$		
e^{at}	$1/(s – a)$	$e^{at}\cos \omega t$	$\frac{(s-a)}{(s-a)^2+\omega^2}$
$\sin \omega t$	$\omega/(s^2 + \omega^2)$		
$\cos \omega t$	$s/(s^2 + \omega^2)$	Additional transforms in pp 264-267/248-251 of Kreyszig 9th/10th edition	
$\sinh \omega t$	$\omega/(s^2 - \omega^2)$		
$\cosh \omega t$	$s/(s^2 - \omega^2)$		

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Review Transforms Properties

- ℒ [$af_1(t) + bf_2(t)$] = aℒ [$f_1(t)$] + bℒ [$f_2(t)$]
- First shifting theorem
- If ℒ[f(t)] = F(s) then ℒ[$e^{at}f(t)$] = F(s – a)
  - Example: ℒ[cos(ωt)] = s/($s^2 + \omega^2$) so ℒ[e^{at}cos(ωt)] = (s – a)/[$(s – a)^2 + \omega^2$]
- Derivative transforms where ℒ[f(t)] = F(s)
  - ℒ[df/dt] = sF(s) – f(0)
  - ℒ[d^2f/dt^2] = s^2F(s) – sf(0) – f'(0)
  - Similar results for higher derivatives

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Solving Differential Equations

- Transform all terms in the differential equation to get an algebraic equation
  - For a differential equation in y(t) we get the transforms Y(s) = ℒ [y(t)]
  - Similar notation for other transformed functions in the equation R(s) = ℒ [r(t)]
- Solve the algebraic equation for Y(s)
- Obtain the inverse transform for Y(s) from tables to get y(t)
  - Manipulations often required to get from Y(s) equation to transforms in tables

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Inverse Transformations

- Use transform table
  - May need partial fractions approach
- Use first shifting theorem discussed last Wednesday
- New methods to be discussed tonight
  - Use second shifting theorem
  - Second shifting theorem depends on definitions of Heavyside unit function and Dirac delta function

Strange Functions

- The Heavyside unit function, u(t – a) is defined to be 0 for t < a and 1 for t ≥ a
  - Represents step from zero to one at x = a
  - Laplace transform is e^{-as}/s
- Delta function, $\delta(x – a)$ is defined such that for any vanishingly small ε, $\delta(x – a)$ = 0 except for $-\varepsilon < x – a < \varepsilon$ and the integral $\int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a)dx = 1$

$$\mathfrak{L}\left[\delta(x-a)\right] = e^{-sa}$$

Second Shifting Theorem

- Applies to $e^{-as}F(s)$, where F(s) is known transform of a function f(t)
- Inverse transform is f(t – a) u(t – a) where u(t – a) is the unit step function
- For $e^{-as}s/(s^2 + \omega^2)$, we have $e^{-as}F(s)$ with $F(s) = s/(s^2 + \omega^2)$ for $f(t) = \cos \omega t$
- Thus $e^{-as}s/(s^2 + \omega^2)$ is the Laplace transform for $\cos[\omega(t – a)]\, u(t – a)$

Review Partial Fractions

- Method to convert fraction with several factors in denominator into sum of individual factors (in denominator)
- Example is F(s) = 1/(s+a)(s+b)
- Write 1/(s+a)(s+b) = A/(s+a) + B/(s+b)
- Multiply by (s+a)(s+b) and equate coefficients of like powers of s
  - 1 = A(s + b) + B(s + a)
  - A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms

Review Partial Fractions II

- A + B = 0 for s^1 terms and 1 = bA + aB for s^0 terms
- Solving for A and B gives A = -B = 1/(b – a)
- Result: 1/(s+a)(s+b) = 1/[(b – a)(s + a)] – 1/[(b – a)(s + b)]
  - So $f(t) = [e^{-at} – e^{-bt}]/(b – a)$
- This actually matches a table entry
- Follow same basic process for more complex fractions
- Special rules for repeated factors and complex factors

Review Partial Fraction Rules

- Repeated fractions for repeated factors

$$\frac{1}{\cdots(s+a)^n\cdots} = \cdots + \frac{A_n}{(s+a)^n} + \frac{A_{n-1}}{(s+a)^{n-1}} + \cdots + \frac{A_2}{(s+a)^2} + \frac{A_1}{s+a} + \cdots$$

- Complex factors $(s + \alpha – i\beta)(s + \alpha + i\beta)$

$$\frac{1}{\cdots(s+\alpha-i\beta)(s+\alpha+i\beta)\cdots} = \cdots + \frac{As+B}{(s+\alpha)^2+\beta^2} + \cdots$$

- Pure imaginary factor is complex factor with $\alpha = 0$

$$\cdots + \frac{1}{s^2+\beta^2} + \cdots = \frac{1}{\cdots(s-i\beta)(s+i\beta)\cdots} = \cdots + \frac{As+B}{s^2+\beta^2} + \cdots$$

Other Applications

- We can apply this to a system of equations for $y_i(t)$
  - Transform all equations from $y_i(t)$ to $Y_i(s)$
  - Solve simultaneous algebraic equations for each $Y_i(s)$
  - Get inverse transforms for $y_i(t)$
  - Sometimes simpler to get some $y_i(t)$ from differential equations after solving one equation using transforms

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System of Equations

- Look at system of two equations from spring-mass system solved previously
  - Have equations for $y_1(t)$ and $y_2(t)$
  - Write $Y_1(s)$ and $Y_2(s)$ for $\mathscr{L}[y_1(t)]$ and $\mathscr{L}[y_2(t)]$
  - Equations from October 11 lecture

$$\frac{d^2y_1}{dt^2}+\frac{k_1+k_2}{m_1}y_1-\frac{k_2}{m_1}y_2=0 \qquad \frac{d^2y_2}{dt^2}-\frac{k_2}{m_2}y_1+\frac{k_3+k_2}{m_2}y_2=0$$

$$s^2Y_1(s)-sy_1(0)-y_1'(0)+\frac{k_1+k_2}{m_1}Y_1(s)-\frac{k_2}{m_1}Y_2(s)=0$$

$$s^2Y_2(s)-sy_2(0)-y_2'(0)-\frac{k_2}{m_2}Y_1(s)+\frac{k_3+k_2}{m_2}Y_2(s)=0$$

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System of Equations II

- Rearrange to show two simultaneous algebraic equations in $Y_1(s)$ and $Y_2(s)$

Eqn A $$\left(s^2+\frac{k_1+k_2}{m_1}\right)Y_1(s)-\frac{k_2}{m_1}Y_2(s)=sy_1(0)+y_1'(0)$$

Eqn B $$-\frac{k_2}{m_2}Y_1(s)+\left(s^2+\frac{k_3+k_2}{m_2}\right)Y_2(s)=sy_2(0)+y_2'(0)$$

- Gauss elimination gives $Y_2(s)$ equation

$$Eqn\ B-Eqn\ A\left[-\frac{k_2}{m_2}\Big/\left(s^2+\frac{k_1+k_2}{m_1}\right)\right]$$

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System of Equations III

$$\left[s^2+\frac{k_3+k_2}{m_2}-\left(\frac{-\frac{k_2}{m_2}}{s^2+\frac{k_1+k_2}{m_1}}\right)\left(-\frac{k_2}{m_1}\right)\right]Y_2(s)=$$

$$sy_2(0)+y_2'(0)-\left(\frac{-\frac{k_2}{m_2}}{s^2+\frac{k_1+k_2}{m_1}}\right)(sy_1(0)+y_1'(0))$$

- Multiply equation by $s^2 + (k_1 + k_2)/m_1$

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System of Equations IV

$$\left[\left(s^2+\frac{k_1+k_2}{m_1}\right)\left(s^2+\frac{k_3+k_2}{m_2}\right)-\left(-\frac{k_2}{m_2}\right)\left(-\frac{k_2}{m_1}\right)\right]Y_2(s)=$$

$$\left(s^2+\frac{k_1+k_2}{m_1}\right)[sy_2(0)+y_2'(0)]-\left(-\frac{k_2}{m_2}\right)(sy_1(0)+y_1'(0))$$

$$\left(s^4+\frac{k_1+k_2}{m_1}s^2+\frac{k_3+k_2}{m_2}s^2+\frac{k_1+k_2}{m_1}\frac{k_3+k_2}{m_2}-\frac{k_2^2}{m_2m_1}\right)Y_2(s)=$$

$$\left(s^2+\frac{k_1+k_2}{m_1}\right)[sy_2(0)+y_2'(0)]+\frac{k_2}{m_2}(sy_1(0)+y_1'(0))$$

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System of Equations V

$$\left[s^4+\frac{k_3+k_2}{m_2}s^2+\frac{k_1+k_2}{m_1}s^2+\frac{k_3+k_2}{m_2}\frac{k_1+k_2}{m_1}-\frac{k_2}{m_2}\frac{k_2}{m_1}\right]Y_2(s)=$$

$$\left(s^2+\frac{k_1+k_2}{m_1}\right)(sy_2(0)+y_2'(0))+\frac{k_2}{m_2}(sy_1(0)+y_1'(0))$$

- The factor for $Y_2(s)$ is the same as the characteristic equation obtained in October 11 lecture

$$\lambda^4+\left(\frac{k_1+k_2}{m_1}+\frac{k_3+k_2}{m_2}\right)\lambda^2+\frac{k_1+k_2}{m_1}\frac{k_3+k_2}{m_2}-\frac{k_2}{m_2}\frac{k_2}{m_1}=0$$

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System of Equations VI

$$\left[s^4+\frac{k_3+k_2}{m_2}s^2+\frac{k_1+k_2}{m_1}s^2+\frac{k_3+k_2}{m_2}\frac{k_1+k_2}{m_1}-\frac{k_2}{m_2}\frac{k_2}{m_1}\right]Y_2(s)=$$

$$\left(s^2+\frac{k_1+k_2}{m_1}\right)(sy_2(0)+y_2'(0))+\frac{k_2}{m_2}(sy_1(0)+y_1'(0))$$

- If all k and all m are the same the equation becomes

$$\left(s^4+\frac{4k}{m}s^2+\frac{3k^2}{m^2}\right)Y_2(s)=$$

$$\left(s^2+\frac{2k}{m}\right)(sy_2(0)+y_2'(0))+\frac{k}{m}(sy_1(0)+y_1'(0))$$

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System of Equations VII

- The term multiplying $Y_2(s)$ can be factored as follows

$$s^4+\frac{4k}{m}s^2+\frac{3k^2}{m^2}=\left(s^2+\frac{3k}{m}\right)\left(s^2+\frac{k}{m}\right)$$

- So the $Y_2(s)$ equation becomes

$$\left(s^2+\frac{3k}{m}\right)\left(s^2+\frac{k}{m}\right)Y_2(s)=$$

$$\left(s^2+\frac{2k}{m}\right)(sy_2(0)+y_2'(0))+\frac{k}{m}(sy_1(0)+y_1'(0))$$

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System of Equations VIII

- Manipulate right side of $Y_2(s)$ equation to combine like powers of s

$$\left(s^2+\frac{3k}{m}\right)\left(s^2+\frac{k}{m}\right)Y_2(s)=$$

$$\left(s^2+\frac{2k}{m}\right)(sy_2(0)+y_2'(0))+\frac{k}{m}(sy_1(0)+y_1'(0))=$$

$$\underbrace{s^3y_2(0)}_{\alpha}+\underbrace{s^2y_2'(0)}_{\beta}+\underbrace{\left(\frac{2k}{m}y_2(0)+\frac{k}{m}y_1(0)\right)}_{\gamma}s+\underbrace{\frac{2k}{m}y_2'(0)+\frac{k}{m}y_1'(0)}_{\delta}$$

$$=\alpha s^3+\beta s^2+\gamma s+\delta$$

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System of Equations IX

- Use partial fractions for $Y_2(s)$; have two pure imaginary factors

$$Y_2(s)=\frac{\alpha s^3+\beta s^2+\gamma s+\delta}{\left(s^2+\frac{3k}{m}\right)\left(s^2+\frac{k}{m}\right)}=\frac{As+B}{s^2+\frac{3k}{m}}+\frac{Cs+D}{s^2+\frac{k}{m}}$$

- Multiply by denominator

$$\alpha s^3+\beta s^2+\gamma s+\delta=(As+B)\left(s^2+\frac{k}{m}\right)+(Cs+D)\left(s^2+\frac{3k}{m}\right)$$

- Expand and equate like powers of s

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System of Equations X

$$\alpha s^3+\beta s^2+\gamma s+\delta=(As+B)\left(s^2+\frac{k}{m}\right)+(Cs+D)\left(s^2+\frac{3k}{m}\right)$$

$$=As^3+A\frac{k}{m}s+Bs^2+B\frac{k}{m}+Cs^3+Cs\frac{3k}{m}+Ds^2+D\frac{3k}{m}$$

$s^3 \quad \alpha=A+C \qquad s^2 \quad \beta=B+D$

$s^1 \quad \gamma=(A+3C)\frac{k}{m} \qquad s^0 \quad \delta=(B+3D)\frac{k}{m}$

$$A=\frac{3\alpha}{2}-\frac{m\gamma}{2k}\quad B=\frac{3\beta}{2}-\frac{m\delta}{2k}\quad C=\frac{m\gamma}{2k}-\frac{\alpha}{2}\quad D=\frac{m\delta}{2k}-\frac{\beta}{2}$$

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System of Equations XI

- The $Y_2(s)$ equation transforms is the sum of transforms for sine and cosine

$$Y_2(s)=\frac{\alpha s^3+\beta s^2+\gamma s+\delta}{\left(s^2+\frac{3k}{m}\right)\left(s^2+\frac{k}{m}\right)}=\frac{As+B}{s^2+\frac{3k}{m}}+\frac{Cs+D}{s^2+\frac{k}{m}}$$

$$F(s)=\frac{As+B}{s^2+\frac{3k}{m}}\Rightarrow f(t)=A\cos\left(t\sqrt{\frac{3k}{m}}\right)+B\sqrt{\frac{m}{3k}}\sin\left(t\sqrt{\frac{3k}{m}}\right)$$

$$F(s)=\frac{Cs+D}{s^2+\frac{k}{m}}\Rightarrow f(t)=C\cos\left(t\sqrt{\frac{k}{m}}\right)+D\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right)$$

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System of Equations XII

- Initial conditions from October 11 lecture
- $y_1(0) = a$, $y_2(0) = -a$, $y_1'(0) = y_2'(0) = 0$

$$\alpha = y_2(0) = -a \qquad \gamma = \frac{2k}{m} y_2(0) + \frac{k}{m} y_1(0) = -\frac{ka}{m}$$

$$\beta = y_2'(0) = 0 \qquad \delta = \frac{2k}{m} y_2'(0) + \frac{k}{m} y_1'(0) = 0$$

- Substitute α, β, γ, and δ values into

$$A = \frac{3\alpha}{2} - \frac{m\gamma}{2k} \quad B = \frac{3\beta}{2} - \frac{m\delta}{2k} \quad C = \frac{m\gamma}{2k} - \frac{\alpha}{2} \quad D = \frac{m\delta}{2k} - \frac{\beta}{2}$$

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System of Equations XIII

$$A = \frac{3\alpha}{2} - \frac{m\gamma}{2k} = -\frac{3a}{2} - \frac{m}{2k}\left(-\frac{ka}{m}\right) = -a \qquad B = \frac{3\beta}{2} - \frac{m\delta}{2k} = 0$$

$$C = \frac{m\gamma}{2k} - \frac{\alpha}{2} = \frac{m}{2k}\left(-\frac{ka}{m}\right) - \frac{-a}{2} = 0 \qquad D = \frac{m\delta}{2k} - \frac{\beta}{2} = 0$$

- Solution with A = –a, B = C = D = 0 is

$$y_2(t) = A\cos\left(t\sqrt{\frac{3k}{m}}\right) + B\sqrt{\frac{m}{3k}}\sin\left(t\sqrt{\frac{3k}{m}}\right)$$

$$+ C\cos\left(t\sqrt{\frac{k}{m}}\right) + D\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right) = -a\cos\left(t\sqrt{\frac{3k}{m}}\right)$$

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System of Equations XIV

- Get y_1 from original differential equation after setting all m and k to be equal

$$\frac{d^2 y_2}{dt^2} - \frac{k_2}{m_2} y_1 + \frac{k_3 + k_2}{m_2} y_2 = 0 \quad \Rightarrow \quad y_1 = \frac{m}{k}\left[\frac{d^2 y_2}{dt^2} + \frac{2k}{m} y_2\right]$$

$$y_1 = \frac{m}{k}\left\{-\frac{3k}{m}\left[-a\cos\left(t\sqrt{\frac{3k}{m}}\right)\right] + \frac{2k}{m}\left[-a\cos\left(t\sqrt{\frac{3k}{m}}\right)\right]\right\}$$

$$y_1(t) = a\cos\left(t\sqrt{\frac{3k}{m}}\right)$$

- y_1 and y_2 same as in October 11 notes

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Other Applications

- Laplace transforms are used to analyze differential equations for control systems
- Define system function or transfer function as $\mathcal{L}$[input] / $\mathcal{L}$[output] for a single input
- Use this function to analyze response to various inputs
- Determine stability of control systems: will a disturbance damp out?

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Laplace Transform Summary

- Use tables to get transforms from y(t) to Y(s) and vice versa
- Differential equation in f(t) and its derivatives becomes algebraic equation in Y(s)
- Solve for Y(s) and rearrange to get terms that you find in transform table
- Use transform table to get y(t) from Y(s)
- Transform method incorporates non-homogenous terms and initial conditions

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Group Exercise

- Form groups of 2-3 people
- Use Laplace transforms to solve the differential equation y'' – 9y = e^{-t} with y(0) = 0 and y'(0) = 2

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Solution to Group Exercise

- Solve y" – 9y = e^{-t} with y(0) = 0 and y'(0) = 2 by Laplace transforms
- Transform differential equation:

$s^2Y(s) - sy(0) - y'(0) - 9Y(s) = 1/(s+1)$

- Substitute initial conditions and solve result for Y(s)

$s^2Y(s) - 0 - 2 - 9Y(s) = 1/(s+1)$

$(s^2 - 9)Y(s) = 2 + 1/(s+1)$

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Solution to Group Exercise II

$(s^2 - 9)Y(s) = 2 + 1/(s+1)$

$$Y(s) = \frac{2}{s^2-9} + \frac{1}{(s^2-9)(s+1)}$$

- Use partial fractions for last term

$$\frac{1}{(s^2-9)(s+1)} = \frac{A}{(s-3)} + \frac{B}{(s+3)} + \frac{C}{(s+1)}$$

$$1 = A(s+1)(s+3) + B(s+1)(s-3) + C(s^2-9)$$

- Set sums of like powers to zero

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Solution to Group Exercise III

$$1 = A(s+1)(s+3) + B(s+1)(s-3) + C(s^2-9)$$

s^2 terms: $0 = A + B + C$

s^1 terms: $0 = 4A - 2B$

s^0 terms: $1 = 3A - 3B - 9C$

- s^1 equation gives B = 2A
- Substituting B = 2A into s^2 equation gives A + 2A + C = 0 or C = –3A
- Substitute B = 2A and C = –3A into s^0 equation to get 1 = 3A – 3(2A) – 9(–3A)

A = 1/24

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Solution to Group Exercise IV

- From A = 1/24 and B = 2A: B = 2/24
- From A = 1/24 and C = -3A: C = -3/24

$$Y(s) = \frac{2}{s^2-9} + \frac{A}{(s-3)} + \frac{B}{(s+3)} + \frac{C}{(s+1)}$$

$$Y(s) = \frac{2}{s^2-9} + \frac{1}{24}\left[\frac{1}{(s-3)} + \frac{2}{(s+3)} - \frac{3}{(s+1)}\right]$$

- From transform table

$$y(t) = \frac{2}{3}\sinh(3t) + \frac{1}{24}[e^{3t} + 2e^{-3t} - 3e^{-t}]$$

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Check Solution for ODE

- Plug solution into original differential equation: y" – 9y = e^{-t}

$$y(t) = \frac{2}{3}\sinh(3t) + \frac{1}{24}[e^{3t} + 2e^{-3t} - 3e^{-t}]$$

$$y'(t) = 2\cosh(3t) + \frac{1}{24}[3e^{3t} - 6e^{-3t} + 3e^{-t}]$$

$$y'' = 6\sinh(3t) + \frac{1}{24}[9e^{3t} + 18e^{-3t} + 3e^{-t}]$$

$$y'' - 9y = 6\sinh(3t) + \frac{1}{24}[9e^{3t} + 18e^{-3t} - 3e^{-t}]$$

$$-9\left[\frac{2}{3}\sinh(3t) + \frac{1}{24}[e^{3t} + 2e^{-3t} - 3e^{-t}]\right] = e^{-t}$$

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Check Boundary Conditions

- Boundary conditions: y(0) = 0; y'(0) = 2

$$y(t) = \frac{2}{3}\sinh(3t) + \frac{1}{24}[e^{3t} + 2e^{-3t} - 3e^{-t}]$$

$$y(0) = \frac{2}{3}\sinh(0) + \frac{1}{24}[e^{0} + 2e^{-0t} - 3e^{-0}]$$

$$= 0 + \frac{1}{24}[1 + 2 - 3] = 0$$

$$y'(t) = 2\cosh(3t) + \frac{1}{24}[3e^{3t} - 6e^{-3t} + 3e^{-t}]$$

$$y'(t) = 2\cosh(0) + \frac{1}{24}[3e^{0} - 6e^{-0} + 3e^{-0}]$$

$$= 2(1) + \frac{1}{24}[3 - 6 + 3] = 2$$

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